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A new look at particle statistics in laser-anemometer measurements

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Many algorithms have been proposed and evaluated for dealing with laser-anemometer measurements in low particle densities since the original paper of McLaughlin & Tiederman (1973). All of them describe the effect of making a measurement whenever a particle is visible to the measurement instrument. However, in many circumstances, one wishes to sample the velocity at a constant rate – for instance, in measurement of velocity fluctuation spectra. It is shown here that the measurement statistics for this case are different from those previously discussed in the literature. The product of the particle density and the sample interval is the controlling parameter for the statistical description of the measurements. The asymptotic forms for low and high particle density-sample time products are derived.

1. Theory and discussion

A laser – Doppler anemometer (LDA) can detect a velocity of a scattering particle when it moves through a prescribed volume of space. If one records a measurement every time a particle passes through the volume, previous authors have shown that the mean number of samples of a velocity v is proportional to ρv, where ρ is the particle density (McLauglin & Tiederman 1973; Buchhave 1975; Durst *et al.* 1976). This can cause a bias in the measured statistics of fluctuating flows. This type of measurement will be referred to here as random measurement. This paper is motivated by a desire to examine other kinds of sampling schemes that may be encountered in laser anemometry. In this paper, schemes that attempt a measurement in a fixed time interval will be examined. Such a scheme could be encountered in a system whose data handling capabilities are saturated by too high a particle appearance rate. In this case, the slowest device in the data handling system would determine the recorded data rate. When this device is saturated, the recorded data are effectively sampled on a regular time interval which is the inverse of the fastest rate the device can handle (Stevenson *et al.* 1980). On the other hand, if velocity fluctuation spectra using a digital Fourier-transform method are desired, the measurements of velocity should be taken on a regular, fixed time interval (Bendat & Piersol 1971). In general, one wants a measurement interval $\Delta\tau$ such that the highest frequency desired is less than $1/(2\Delta\tau)$. George & Lumley (1973) have shown that for laser anemometers, the spectrum is obscured by 'finite transit time noise' for frequencies above the inverse of the time it takes a particle to travel through the measurement volume.

Therefore, in this paper it will be assumed that time is divided into equal length measurement cycle times of length $\Delta\tau$, where $\Delta\tau$ is small compared to the velocity fluctuation correlation time, but larger than the mean time required for a particle to travel through the volume. The conditions where the cycle time is long compared to the correlation time of the flow as discussed by Barnett & Bentley (1974), are a special case of random measurement and will not be discussed here.

It will now be shown how the measurement statistics for regular sampling differs from those for random measurement. A one-dimensional flow system will be considered; however, the calculations can be generalized to three dimensions in a straightforward, if tedious, fashion.

Let L be the length of the sample region. A particle must traverse it completely in order to be measured. Consider a processor that attempts one velocity measurement every time interval $\Delta\tau$. It is assumed that it successfully gets a measurement if any number of particles pass through the sample region during $\Delta\tau$. We will assume that it stores only the first successfully measured velocity during each cycle.

The particles have a Poisson distribution, so that the expected number of particles in a volume L' is $\rho L'$ (Feller 1957). The probability of there being at least one particle in a volume L' is $1-\exp[-\rho L']$. The available volume to be sampled during each measurement is $v\Delta\tau - L$, since a particle must traverse L before it is measured. The probability that there was at least one particle present in that volume is

$$1-\exp[-\rho(|v|\,\Delta\tau - L)]. \tag{1}$$

The probability of measuring a velocity v during any one $\Delta\tau$ is thus

$$\begin{array}{ll} P(v)\,(1-\exp[-\rho(|v|\,\Delta\tau - L)]\,dv, & |v| \geqslant \dfrac{L}{\Delta\tau}; \\ 0, & 0 \leqslant |v| < L/\Delta\tau. \end{array} \tag{2}$$

$P(v)$ is the velocity probability density function. The two regions of v reflect the fact that no measurement is made if the particle does not cross at least a volume L. The probability density merely shows the fact that the larger the volume swept through the sample region L, the higher the probability of finding at least one particle in it. Note that it predicts a decreased bias in the limit $\rho \to \infty$, independent of the turbulence field described by $P(v)$, since the probability density tends to just $P(v)\,dv$ in this limit. Further, the error tends to zero as the ratio of the velocity to the minimum measurable velocity, $L/\Delta\tau$, increases.

Assuming all unsuccessful measurement attempts are dropped, the mean measured velocity $\bar{v}_m$ is given by

$$\bar{v}_m = \frac{\displaystyle\int_{L/\Delta\tau}^{\infty} vP(v)\,(1-\exp[-\rho(v\Delta\tau - L)])\,dv + \int_{-\infty}^{-L/\Delta\tau} vP(v)\,(1-\exp[\rho(v\Delta\tau + L)])\,dv}{\displaystyle\int_{L/\Delta\tau}^{\infty} P(v)\,(1-\exp[-\rho(v\Delta\tau - L)])\,dv + \int_{-\infty}^{-L/\Delta\tau} P(v)\,(1-\exp[\rho(v\Delta\tau + L)])\,dv.} \tag{3}$$

The denominator of this term is the probability of getting any measurement in time $\Delta\tau$, and is denoted here as the efficiency factor Ψ.

The efficiency and thus the corrections from the mean velocity and turbulence intensity are a function of the three dimensionless parameters: (*a*) ρL, the mean number of particles in the minimum sample volume; (*b*) $\bar{v}\Delta\tau/L$ the ratio of the average velocity to the minimum measurable velocity, denoted here the velocity ratio; and (*c*) $\sigma_v/\bar{v}$, the turbulence intensity.

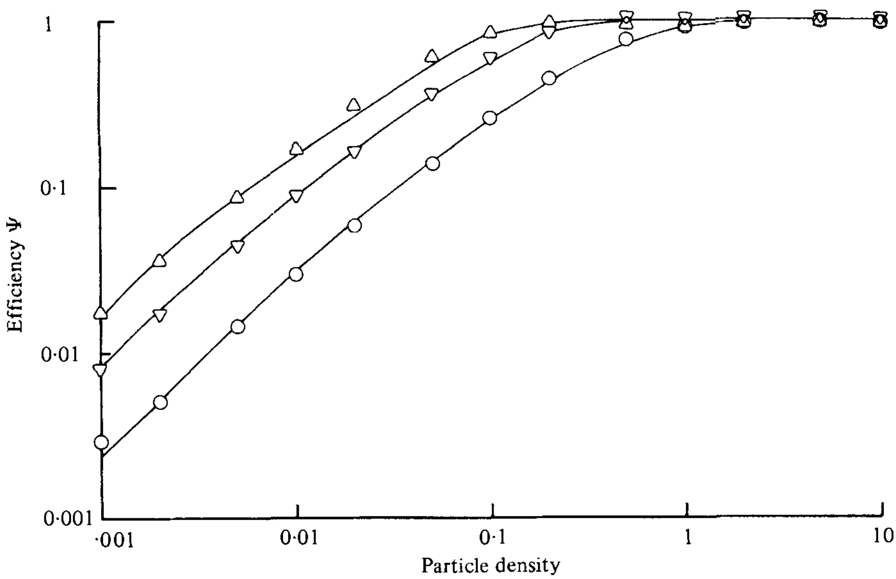

FIGURE 1. Efficiency versus particle density at three different velocity ratios for a detector measuring any number of particles in the volume. At low particle densities, the efficiency is approximately the particle density times the velocity ratio minus one. The efficiency asymptotically approaches one as the particle density goes to infinity. Velocity ratios: ○, 4; △, 10; ▽, 20.

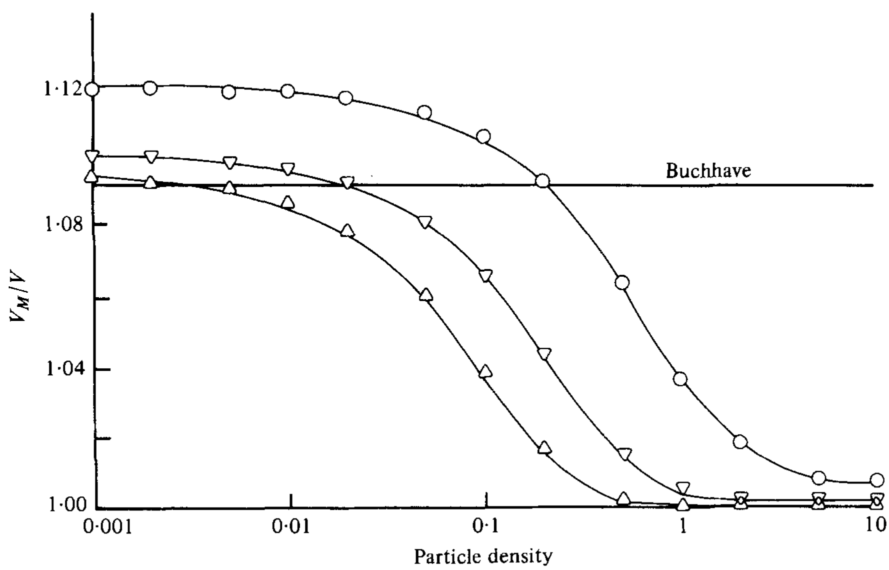

FIGURE 2. The ratio of the mean measured velocity to the actual mean velocity as a function of particle density for a turbulence intensity of 0·3 using a multi-particle detector. Three different velocity ratios are shown: the errors decrease as the particle density increases. The ratio for a random measurement of velocity under these conditions is predicted by Buchhave to be 1·09. Velocity ratios: ○, 4; ▽, 10; △, 20.

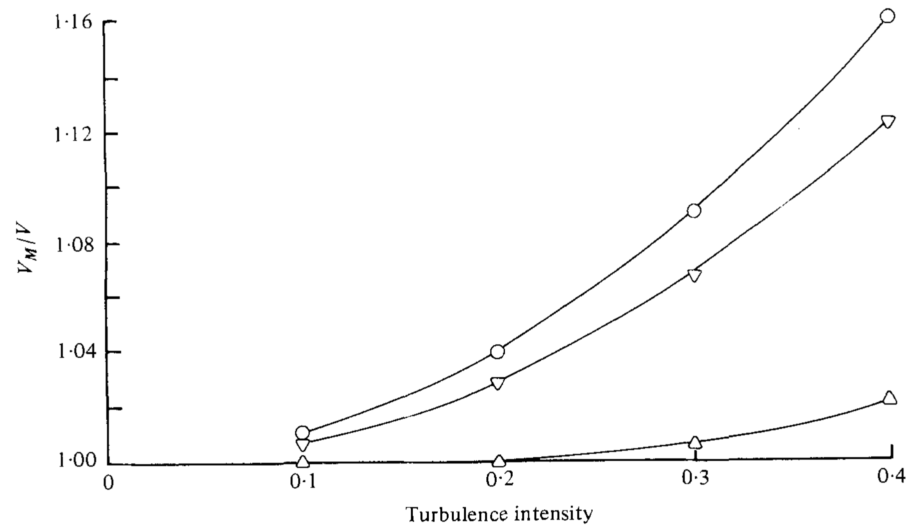

FIGURE 3. The ratio of the measured mean velocity to the actual mean velocity as a function of turbulence intensity for a multiparticle processor. A velocity ratio of ten was used. Two different particle densities are shown along with Buchhave's prediction for comparison. Again, note the strong dependence of the error on the particle density. Velocity ratio = 10. ○, Buchhave; ▽, ρL = 0·1; △, ρL = 1.

To demonstrate the form of (3), we have assumed a velocity probability of the form

$$P(v) = (1/\pi 2\sigma_v^2)^{\frac{1}{2}} \exp\left[-\frac{(v-\bar{v})^2}{2\sigma_v^2}\right].$$

The measured turbulence intensities are computed using the same probability function. The intensities are computed as deviations from the measured mean velocity.

Figure 1 shows a plot of the efficiency ψ, as a function of the velocity ratio and mean particle density. The efficiency is a very weak function of the turbulence intensity and this dependency is not shown. Note that the efficiency tends to one, near a particle density of one per sample volume. The efficiency decreases as the velocity ratio decreases. This is not surprising, since decreasing the velocity ratio is equivalent to sampling a smaller volume of space for a particle.

Figure 2 shows the ratio of the measured mean velocity to the actual mean velocity as a function of particle density and velocity ratio at a turbulence intensity $(\sigma_v/\bar{v})$ of 0·3. The error decreases as the particle density increases: it also decreases as the velocity ratio increases. Buchhave (1975) starting with Tiedermann & McLaughlin's (1973) theory, calculated a constant error ratio of $1+(\sigma_v/\bar{v})^2$ if one measures every particle. This is shown as the horizontal line on the plot. Note the error is generally larger at low concentrations than Buchhave predicts. This is the effect of the 'dead zone' of unmeasurable velocities where the velocity cannot carry a particle through the volume in time $\Delta\tau$. In the limit of infinite particle density, the errors do not go to zero, keeping a small value due to the dead zone.

Figure 3 shows a plot of the mean velocity error, $\bar{v}_m/\bar{v}$, as a function of turbulence intensity for particle densities of 0·1 and 1 at a velocity ratio of 10. Again, Buchhave's correction factor is plotted for comparison. For low particle densities, the correction

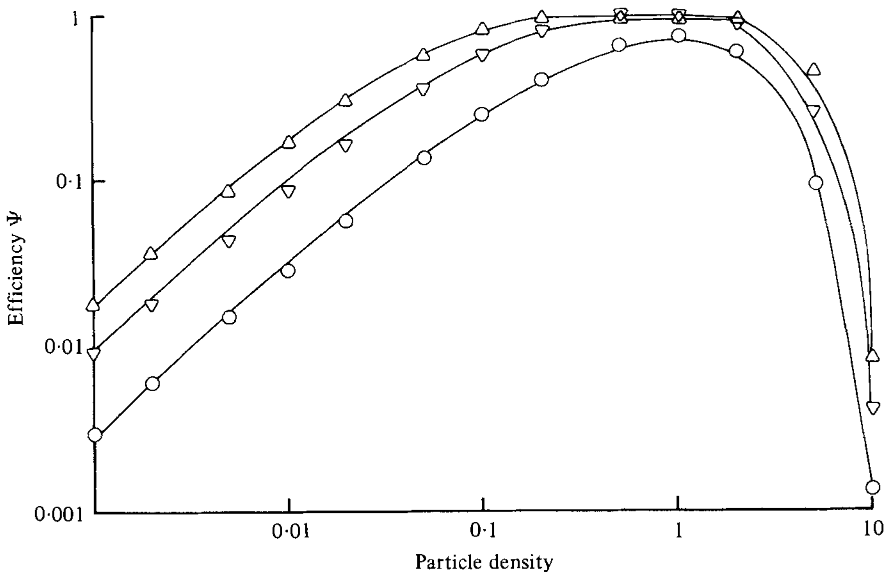

FIGURE 4. The approximate form of the efficiency versus particle density for a counter type detector with a verifier circuit. Three different velocity ratios are used. The efficiency peaks near a particle density of one for each case. Velocity ratios: ○, 4; ▽, 10; △, 20.

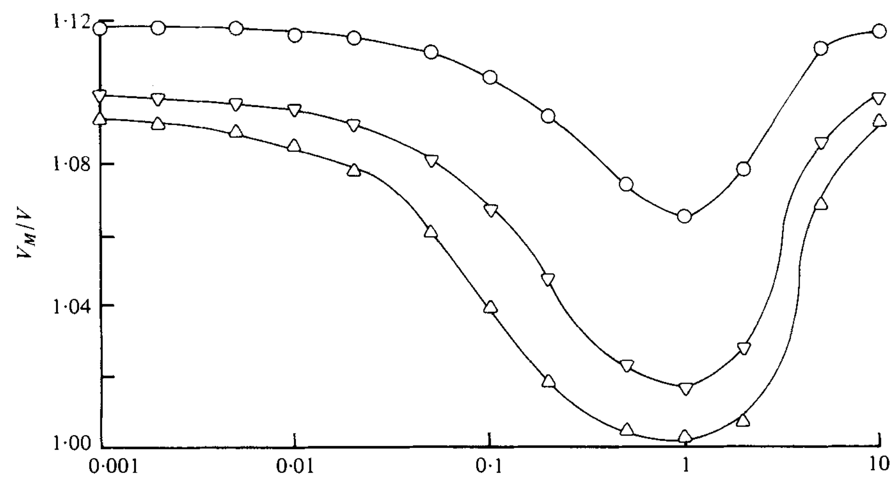

FIGURE 5. The approximate ratio of the mean-measured velocity to the actual mean velocity as a function of particle density for a counter detector with a verifier. The error is a minimum for all velocity ratios near a particle density of one per volume. $\sigma_v/V = 0{\cdot}3$. Velocity ratios: ○, 4; ▽, 10; △, 20.

is close to the random measurement prediction. At high particle concentrations, the errors are much smaller.

2. Validation circuits

If a counter is used with a validation circuit that rejects a measurement when more than one particle is present in the volume L at one time, the probability function has to be modified. Actually, these circuits do not reject all measurements from multi-particle systems, only those that cause a detectable change in the number of zero crossings from one part of the measurement to another. The exact expression for the probability of a successful measurement is complicated, depending partly on the phase error tolerance built into the system and the noise level. Therefore, here, we will not derive an exact form from the probability function, but an approximation that has the correct qualitative dependence on the parameters and the correct quantitative behaviour at the limits.

The probability of finding exactly one particle in a volume L for a particle density ρ is $\rho L \exp[-\rho L)$ (Feller 1957). The probability of finding a volume with exactly one particle in n independent measurements is (Feller 1957) $1-(1-\rho L' \exp[-\rho L'])^n$. We will approximate the number of independent samples per measurement time $\Delta\tau$ by $n = (v\Delta\tau - L)/L$.

The probability density of a measurement of velocity v is thus

$$P(v)\left(1-(1-L\exp[-\rho L])^{(|v|\Delta\tau/L)}\right) \quad \text{if} \quad \frac{|v|\Delta\tau}{L} \geqslant 1,$$

and zero otherwise. This tends to the same limit for the particle density going to zero or to infinity. In both cases, the probability tends to zero. However, it peaks at $\rho L = 1$, to a value of

$$\left(1-(0{\cdot}632)^{(v\Delta\tau/L-1)}\right) \times P(v).$$

Clearly, the best efficiencies will be obtained when $\rho L = 1$ and the velocity ratio is the greatest.

Figure 4 is a plot of the efficiency for counters with validation circuits (ψ_0) versus particle density for three different velocity ratios. Again, the efficiency is a weak function of the turbulence intensity. Initially, the efficiencies increase with increasing particle densities reaching a peak at a particle density of one per sample volume and then decrease precipitously for larger particle densities. The probability for more than one particle being present, rises sharply above a mean of one particle per volume. The validation circuit rejects the measurements when more than one particle is present, thus the overall measurement efficiency drops. Again, this results in an increase in statistical bias.

Figure 5 shows a plot of the ratio of the measured mean velocity to the actual mean velocity versus particle density at a turbulence intensity of 0·3 at various velocity ratios. The measured velocity errors decrease as the particle density increases up to a density of one; above that density, the errors increase again. The magnitude of the error can be decreased by increasing the velocity ratio.

			Turbulence intensity							
			0·1		0·2		0·3		0·4	
Velocity ratio	Particle density	Eff.	*vc*	*ic*	*vc*	*ic*	*vc*	*ic*	*vc*	*ic*
4	0·001	0·003	1·013	0·978	1·053	0·915	1·119	0·823	1·205	0·729
4	0·002	0·005	1·013	0·978	1·053	0·915	1·119	0·823	1·205	0·730
4	0·005	0·014	1·013	0·978	1·053	0·915	1·118	0·823	1·204	0·730
4	0·010	0·029	1·013	0·978	1·053	0·915	1·118	0·824	1·203	0·730
4	0·020	0·058	1·013	0·978	1·052	0·916	1·116	0·825	1·200	0·731
4	0·050	0·139	1·012	0·978	1·050	0·917	1·112	0·827	1·193	0·734
4	0·100	0·259	1·011	0·979	1·046	0·920	1·105	0·831	1·182	0·739
4	0·200	0·451	1·010	0·981	1·040	0·926	1·092	0·839	1·163	0·748
4	0·500	0·777	1·006	0·986	1·025	0·942	1·064	0·862	1·119	0·774
4	1·000	0·950	1·002	0·992	1·012	0·963	1·037	0·894	1·080	0·810
4	2·000	0·997	1·000	0·998	1·003	0·986	1·018	0·931	1·052	0·851
4	5·000	1·000	1·000	0·999	1·000	0·997	1·008	0·959	1·036	0·886
4	10·000	1·000	1·000	0·999	1·000	0·998	1·007	0·966	1·031	0·898
10	0·001	0·008	1·011	0·982	1·044	0·933	1·099	0·858	1·174	0·774
10	0·002	0·017	1·011	0·982	1·044	0·933	1·099	0·858	1·173	0·774
10	0·005	0·044	1·011	0·983	1·043	0·934	1·098	0·859	1·171	0·775
10	0·010	0·086	1·011	0·983	1·043	0·935	1·096	0·860	1·168	0·777
10	0·020	0·165	1·010	0·983	1·041	0·936	1·092	0·863	1·162	0·780
10	0·050	0·362	1·009	0·985	1·036	0·941	1·081	0·871	1·145	0·790
10	0·100	0·593	1·007	0·987	1·028	0·948	1·066	0·883	1·121	0·805
10	0·200	0·835	1·004	0·991	1·017	0·961	1·044	0·905	1·086	0·834
10	0·500	0·989	1·001	0·997	1·004	0·986	1·015	0·950	1·040	0·891
10	1·000	1·00	1·000	0·999	1·000	0·997	1·005	0·976	1·022	0·928
10	2·000	1·000	1·000	0·999	1·000	0·999	1·002	0·987	1·015	0·948
10	5·000	1·000	1·000	0·999	1·000	0·999	1·002	0·991	1·011	0·960
10	10·000	1·000	1·000	0·999	1·000	0·999	1·001	0·992	1·010	0·963
20	0·001	0·018	1·010	0·984	1·042	0·938	1·094	0·867	1·164	0·788
20	0·002	0·037	1·010	0·984	1·041	0·938	1·093	0·868	1·163	0·788
20	0·005	0·090	1·010	0·984	1·040	0·939	1·091	0·870	1·159	0·791
20	0·010	0·173	0·010	0·984	1·038	0·941	1·087	0·872	1·153	0·794
20	0·020	0·316	1·009	0·985	1·035	0·944	1·079	0·878	1·141	0·801
20	0·050	0·613	1·006	0·988	1·026	0·953	1·061	0·893	1·111	0·822
20	0·100	0·850	1·004	0·992	1·015	0·966	1·039	0·916	1·076	0·852
20	0·200	0·978	1·001	0·996	1·005	0·984	1·017	0·950	1·041	0·896
20	0·500	1·000	1·000	0·999	1·000	0·998	1·003	0·984	1·016	0·947
20	1·000	1·000	1·000	0·999	1·000	0·999	1·001	0·992	1·009	0·966
20	2·000	1·000	1·000	0·999	1·000	0·999	1·001	0·994	1·007	0·975
20	5·000	1·000	1·000	0·999	1·000	0·999	1·001	0·996	1·005	0·980
20	10·000	1·000	1·000	0·999	1·000	0·999	1·001	0·996	1·005	0·982

$vc = v_m/\bar{v}$; $ic = \sigma_m\bar{v}/\sigma_v v_m$.

TABLE 1

A compilation of the correction factors for various conditions is given in table 1 (detector without verifier) and table 2 (detector with verifier).

3. Conclusions

The statistics of velocity measurements of fluctuating flows obtained from laser anemometers using counter-type detectors are biased towards higher velocities. This bias is due to the fact that it is more probable that the detector sees a particle of higher

			Turbulence intensity							
			0·1		0·2		0·3		0·4	
Velocity ratio	Particle density	Eff.	*vc*	*ic*	*vc*	*ic*	*vc*	*ic*	*vc*	*ic*
4	0·001	0·002	1·013	0·978	1·053	0·915	1·119	0·823	1·206	0·724
4	0·002	0·005	1·013	0·978	1·053	0·915	1·119	0·823	1·206	0·724
4	0·005	0·014	1·013	0·978	1·053	0·915	1·118	0·823	1·205	0·724
4	0·010	0·029	1·013	0·978	1·053	0·915	1·118	0·824	1·024	0·724
4	0·020	0·057	1·013	0·978	1·052	0·916	1·116	0·825	1·202	0·725
4	0·050	0·136	1·012	0·978	1·050	0·917	1·112	0·827	1·195	0·727
4	0·100	0·248	1·012	0·979	1·047	0·920	1·105	0·831	1·185	0·731
4	0·200	0·415	1·010	0·981	1·041	0·925	1·095	0·838	1·168	0·737
4	0·500	0·662	1·007	0·984	1·031	0·934	1·075	0·852	1·140	0·747
4	1·000	0·747	1·006	0·985	1·027	0·939	1·067	0·859	1·128	0·750
4	2·000	0·612	1·008	0·983	1·034	0·932	1·080	0·849	1·146	0·745
4	5·000	0·097	1·013	0·978	1·051	0·917	1·114	0·826	1·198	0·726
4	10·000	0·001	1·013	0·978	1·053	0·915	1·119	0·823	1·206	0·724
10	0·001	0·008	1·011	0·982	1·044	0·933	1·099	0·858	1·176	0·764
10	0·002	0·017	1·011	0·982	1·044	0·933	1·099	0·858	1·175	0·765
10	0·005	0·043	1·011	0·983	1·043	0·934	1·098	0·859	1·173	0·766
10	0·010	0·085	1·011	0·983	1·043	0·935	1·096	0·860	1·170	0·767
10	0·020	0·163	1·010	0·983	1·041	0·936	1·092	0·862	1·164	0·770
10	0·050	0·355	1·009	0·985	1·036	0·941	1·082	0·870	1·148	0·777
10	0·100	0·574	1·007	0·987	1·029	0·947	1·068	0·881	1·126	0·788
10	0·200	0·800	0·005	0·990	1·019	0·959	1·048	0·900	1·097	0·803
10	0·500	0·961	1·002	0·995	1·008	0·978	1·024	0·932	1·066	0·810
10	1·000	0·984	1·001	0·997	1·005	0·984	1·017	0·944	1·062	0·793
10	2·000	0·942	1·002	0·994	1·010	0·974	1·028	0·925	1·071	0·813
10	5·000	0·265	1·009	0·984	1·038	0·938	1·087	0·866	1·156	0·773
10	10·000	0·004	1·011	0·982	1·044	0·933	1·100	0·857	1·176	0·764
20	0·001	0·018	1·010	0·984	1·042	0·938	1·094	0·867	1·167	0·777
20	0·002	0·037	1·010	0·984	1·041	0·938	1·093	0·868	1·165	0·778
20	0·005	0·090	1·010	0·984	1·040	0·939	1·091	0·869	1·162	0·780
20	0·010	0·172	1·010	0·984	1·038	0·941	1·087	0·872	1·155	0·783
20	0·020	0·313	1·009	0·985	1·035	0·944	1·080	0·877	1·144	0·788
20	0·050	0·604	1·006	0·988	1·026	0·953	1·062	0·892	1·116	0·804
20	0·100	0·835	1·004	0·991	1·016	0·965	1·041	0·913	1·085	0·823
20	0·200	0·966	1·001	0·996	1·006	0·981	1·020	0·942	1·057	0·836
20	0·500	0·999	1·000	0·999	1·001	0·996	1·007	0·971	1·066	0·700
20	1·000	1·000	1·000	0·999	1·000	0·998	1·005	0·975	1·118	0·196
20	2·000	0·997	1·000	0·999	1·001	0·994	1·008	0·967	1·056	0·769
20	5·000	0·479	1·007	0·987	1·030	0·948	1·070	0·885	1·129	0·796
20	10.000	0·008	1·010	0·984	1·042	0·938	1·094	0·867	1·167	0·777

$vc = v_m/\bar{v}$; $ic = \sigma_v v_m$.

TABLE 2

velocity than one of lower velocity. If the detector attempts to make measurements in regular intervals, this statistical bias is a function of the particle density – in strong contrast to the predicted behaviour of systems that make a measurement for every particle the detector sees. For the type of detectors discussed here, the statistical bias is a minimum when the detector obtains a measurement every sample period. For many practical situations, the bias can be made negligibly small.

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